Dynamical anomalies and the role of initial conditions in the HMF model

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Abstract

We discuss the role of the initial conditions for the dynamical anomalies observed in the quasi-stationary states of the Hamiltonian Mean Field (HMF) model.

Key words: Hamiltonian dynamics; Long-range interactions; Power-law correlations; Anomalous diffusion; Tsallis thermostatistics.
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1 Introduction

The Hamiltonian Mean Field (HMF) model is a system of N fully-coupled inertial spins which has been intensively studied in the last years [1,2,3,4,5,6,7,8,9,10,11,12]. The model is particularly important for the paradigmatic anomalous behavior exhibited by its out-of-equilibrium dynamics. Motivated by recent papers [13,14] in which such anomalies were conjectured to exist only for very special initial conditions, in this paper, we show that anomalous dynamics, and in particular fractal-like structures in phase space, power-law decay of correlations and superdiffusion are obtained for a large class of initial conditions. Our results indicate that these anomalous behavior represents more the rule rather than the exception. Connections with Tsallis thermostatistics [15,16,17,18] are briefly addressed.

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2 The model

The model describes a system of N planar classical spins $\vec{s_i} = (cos\theta_i, sin\theta_i)$ with unitary mass and with an infinite-range interaction [1]. The Hamiltonian, in the ferromagnetic case, can be written as

$$H = K + V = \sum_{i=1}^{N} \frac{{p_i}^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - cos(\theta_i - \theta_j)] \quad , \tag{1}$$

where $\theta_i \in [-\pi, \pi]$, is the angle of the *ith* spin and p_i the corresponding conjugate variable representing the rotational velocity. Since the modulus of each spin is unitary, we can also view the system as N interacting particles moving on the unit circle. The standard order parameter of the model is the magnetization M, defined as $M = \frac{1}{N} | \sum_{i=1}^{N} \vec{s_i} \ |$. The equilibrium solution of the model exhibits a second-order phase transition from a low-energy condensed (ferromagnetic) phase with magnetization $M \neq 0$, to a high-energy (paramagnetic) one, where the spins are homogeneously oriented on the unit circle and $M = 0$. The *caloric curve*, i.e. the dependence of the energy density $U = E/N$ on the temperature T, is given by $U = \frac{T}{2} + \frac{1}{2}(1 - M^2)$ [1,2]. The critical point is at energy density $U_c = \frac{3}{4}$, which corresponds to a critical temperature $T_c = \frac{1}{2}$.
The dynamics of the HMF model shows several anomalies before complete equilibration. More precisely, if we adopt the so-called $M1$ initial conditions, i.e. $\theta_i = 0$ for all i ($M(0) = 1$) and velocities uniformly distributed (*water bag*), the results of the simulations, in a special region of energy values ($\frac{1}{2} < U < U_c$), show a disagreement with the equilibrium prediction for a transient regime whose lifetime depends on the system size N [4,6]. In such a regime, the system remains trapped in metastable quasi-stationary states (QSS) characterized by a temperature lower than the equilibrium one, a vanishing value of magnetization and Lyapunov exponents [5], anomalous diffusion and long-range correlations [3,4,8], very slow and glassy-like dynamics [7,9,10].

3 Out-of-equilibrium dynamics vs initial conditions

In this Section we show, by means of a series of numerical simulations, that the majority of the dynamical anomalies of the QSS regime are present not only for M1 initial conditions (ic), but also when the initial magnetization $M(t = 0)$ is in the range $(0, 1]$. We concentrate on the energy value $U = 0.69$, where the anomalies are more evident. The case M(0)=0 has been studied

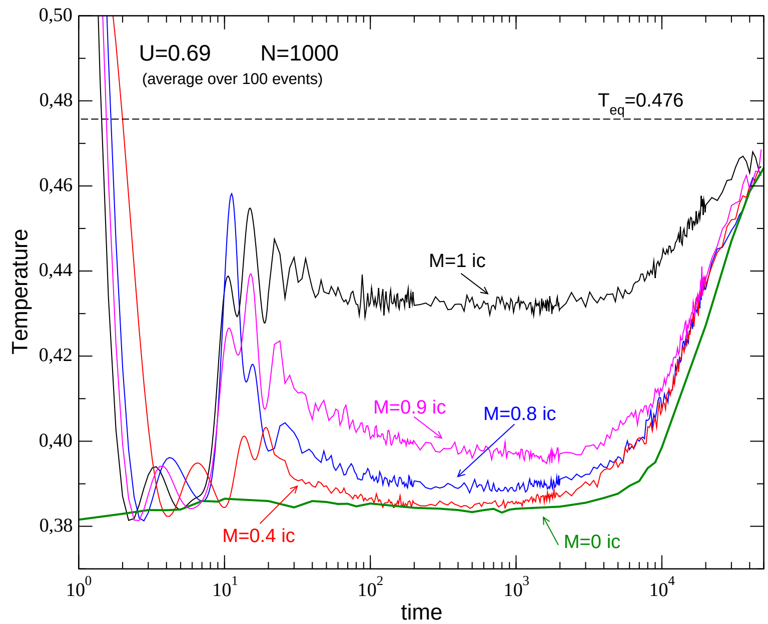

Fig. 1. Time evolution of the temperature for $U = 0.69$, $N = 1000$. Different initial conditions of magnetization, ranging from $M = 1$ to $M = 0$, averaged over 100 events, are considered.

in Refs. [8,13,14] and corresponds to a stable stationary state of the Vlasov equation [14]. Such a state is spatially homogeneous from the beginning, thus the force acting on each spin is zero since $t = 0$ and correlations are almost absent. This case represents a limiting situation as it will be shown in the following. To prepare the initial magnetization in the range $0 \leq M \leq 1$, we distribute uniformly the orientation angle of the spins into a variable portion of the unitary circle. In this way we fix the potential energy and we assign the remaining part of the total energy as kinetic energy by using the usual water bag uniform distribution for the momenta. For the details on the numerical integration used see refs. [2,3,5].

3.1 Temperature plateaux and structures in phase-space

In fig.1 we plot the time evolution of the temperature T, calculated by means of the average kinetic energy as $T = 2 < K > /N$. The simulations refer to $U = 0.69$, $N = 1000$ and to different initial conditions. All the curves, except the one referring to M=0 ic, show a fast relaxation from the high initial temperature value. Then one observes small fluctuations around a plateau region, before relaxation to equilibrium. We report as dashed line the equilibrium value. The curves are obtained after an average over 100 events. Increasing this number fluctuations disappears completely. The plateaux temperatures are smaller than the equilibrium temperature and depend on the size and on the initial conditions used. No qualitative difference is found for greater sizes,

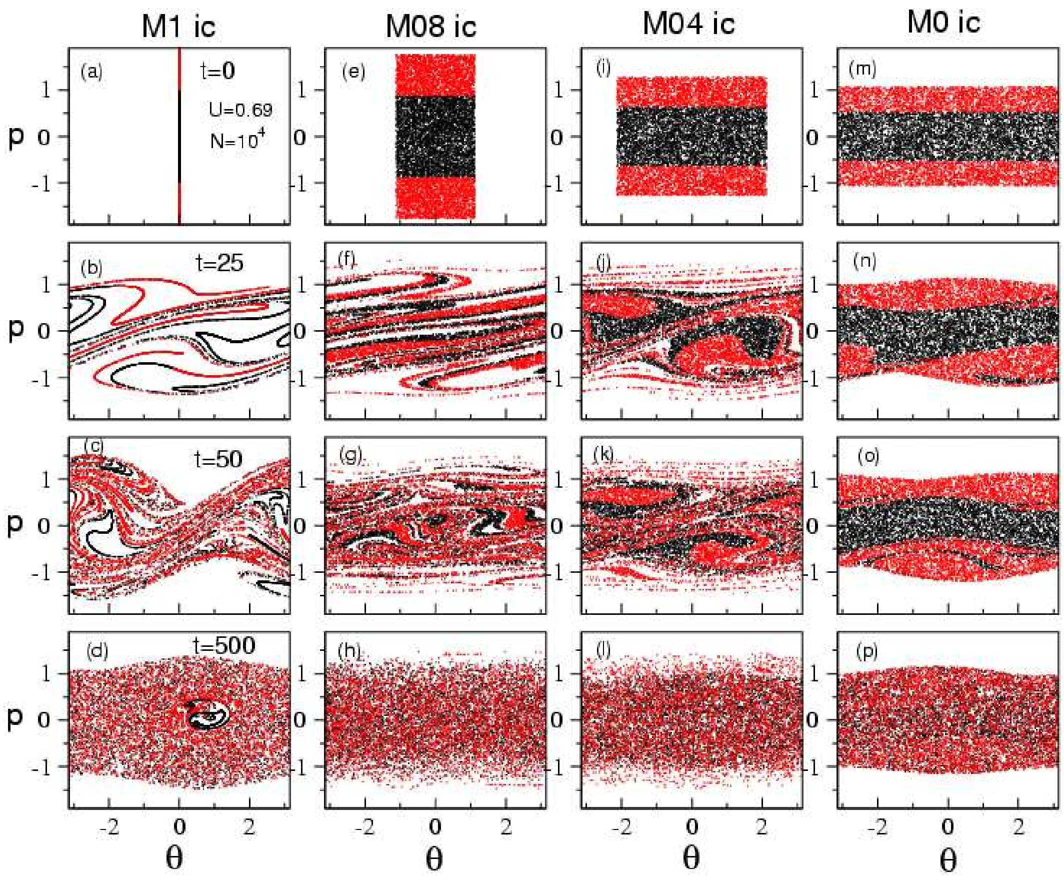

Fig. 2. For $U = 0.69$, $N = 10000$ we show a sequence of snapshots of the μ-space at different times (from top to bottom). Four different initial magnetizations, namely $M = 1, 0.8, 0.4, 0,$ are considered.

apart from the fact that for $N \to \infty$ all the plateaux increase in time duration and tend to a limiting temperature $T_{N=\infty} = 0.38$ (for $U = 0.69$). For any finite N the system always relaxes to the usual Boltzmann-Gibbs equilibrium value $T_{eq} = 0.476$, although the relaxation time diverges linearly with N. Therefore the QSS regime can be considered a real equilibrium regime when the infinite size limit is taken before the infinite time limit [4,6].
Although from the plot of T vs time the QSS behavior seems to be the same for all the initial conditions, this is not true for what concerns the correlations and their decay. In fig.2, for $U = 0.69$ and $N = 10000$, we show the μ-space of the system at different times and for four different initial magnetizations. The different colours distinguish fast initial spin velocities (in red) and slow ones (in black). While in the QSS regime the magnetization vanishes immediatly after the violent initial relaxation, dynamical structures, having fractal-like features [4], emerge in the μ-space and then fade away only after a long time. These structures, that have been already studied in detail for M1 ic [4,8], depend on the initial magnetization and thus on the initial value of the force, acting on each spin j, being $F_j = -M_x sin(\theta_j) + M_y cos(\theta_j)$. They are clearly visible for M08 and also for M04, while they are almost absent for M0 ic.

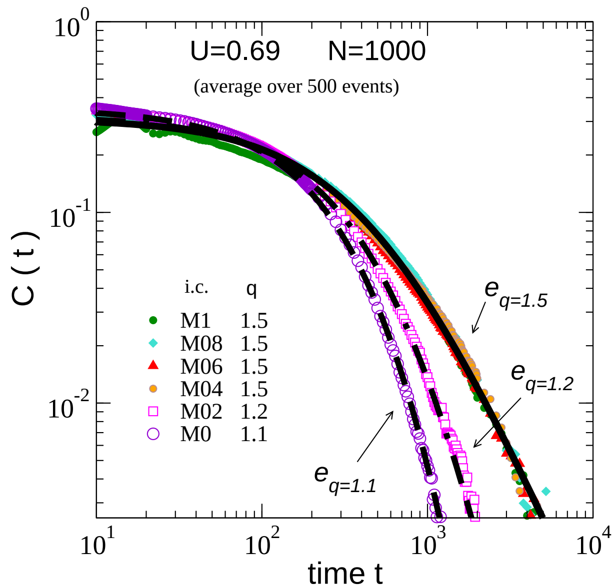

Fig. 3. Correlation functions vs time for different initial magnetizations (symbols). The curves are normalized q-exponentials defined by eq.(3).

3.2 Long-range correlations and anomalous diffusion

In this section we focus on the decay of velocity correlations. A quantitative way to estimate the velocity correlations is the autocorrelation function, defined as

$$C(t) = \frac{1}{N}\sum_{j=1}^{N} p_j(t)p_j(0) \quad . \tag{2}$$

In fig.3 we plot the velocity autocorrelation functions for $N = 1000$ and $M(0) = 1, 0.8, 0.6, 0.4, 0.2, 0$. Averages are taken over 500 different realizations. The initial fast relaxation illustrated in fig.1 has not been considered. For $M(0) \geq 0.4$ the correlation functions are very similar, while the decay is faster for $M(0) = 0.2$ and even more for $M(0) = 0$. The autocorrelation functions can be reproduced by means of the q-exponential function

$$e_q(z) = [1 + (1-q)z]^{\frac{1}{(1-q)}} \tag{3}$$

proposed by Tsallis in his generalized thermodynamics [15,16,17] with $z = -\frac{x}{\tau}$. Here τ is a characteristic time. Notice that one recovers the usual exponential decay for $q = 1$ [15,16,17]. In this way we can quantitatively discriminate between the different initial conditions. In fact we get a q-exponential with $q = 1.5$ for $M \geq 0.4$, while we get $q = 1.2$ and $q = 1.1$ for $M = 0.2$ and for $M = 0$ respectively. Thus for $M > 0$ correlations exhibit a long-range nature and a slow decay: they are very similar, but they diminish progressively

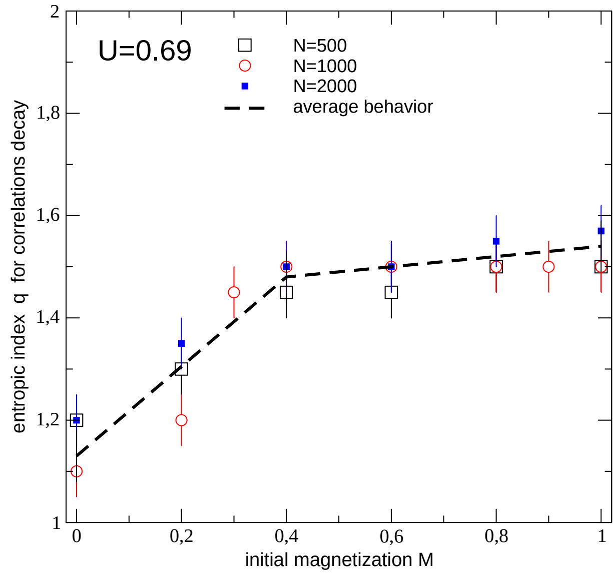

Fig. 4. Entropic index q extracted from the decay of the correlation functions as a function of the initial magnetization and for several system sizes. The dashed line is the average behavior.

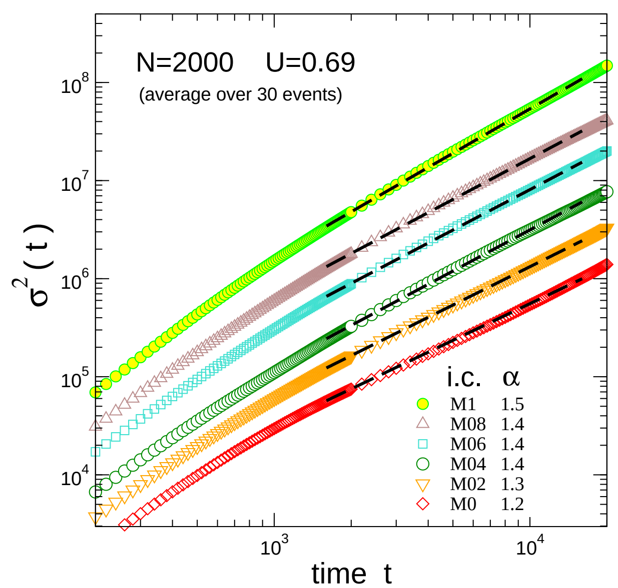

Fig. 5. We plot the mean square displacement of the angular motion $\sigma^2 \propto t^\alpha$ [3] vs time for different initial magnetizations. The exponent α which characterizes the behavior in the QSS regime and in its successive decay is also reported. The dashed lines have a slope corresponding to these values.

below $M = 0.4$ to become almost exponential for $M = 0$. In ref. [13] it was shown that this limiting case can been also fitted by a stretched exponential. In fig.4 we plot the q-values obtained for different initial conditions and different sizes of the system. It is possible to observe that the increase of q from 1.1 to 1.5 with the initial magnetization and the almost constant value for $M(0) > 0.4$. is not much dependent on the size of the system. Thus, in

general, long-range correlations are obtained for a wide spectrum of initial conditions, while, again, the case M0 seems to be a very special one. The latter has been studied in detail in refs [13,14] and it has been proven to be a stationary solution of the Vlasov equation that tends to attract the QSS. On the other hand, Tsallis nonextensive thermostatistics scenario seems to be a better candidate to explain the dynamical anomalies observed for finite initial magnetization [6,16,17,18]. A further indication in this direction is provided by the correlation between the value of the entropic index q and the exponent α of anomalous diffusion [19,20]. The latter occurs if the mean square displacement in angle $\sigma^2 \propto t^\alpha$ has an exponent $\alpha \neq 1$. Superdiffusion ($\alpha > 1$) has been observed in the HMF model for M1 initial conditions [3,6]. In the present investigation we have checked that even decreasing the initial magnetization the system continues to show superdiffusion. We illustrate this behavior in fig.5, where one sees, after a ballistic regime ($\alpha = 2$) proper of the initial fast relaxation, that in the QSS plateau region and afterwards, the system shows superdiffusion. The exponent goes progressively from $\alpha = 1.4 - 1.5$ for $0.4 < M(0) < 1$ to $\alpha = 1.2$ for M0. In the latter case we have checked that increasing the size of the system diffusion tends to be normal ($\alpha = 1$ for N=10000). In correspondence, as previously shown, the entropic index q characterizing the correlation decay, diminishes from 1.5 to 1.1, in good agreement with the relationship $\alpha = 2/(3-q)$ [19]. A detailed study of this behavior will be reported elsewhere [21].

4 Conclusions

The results discussed in this paper show that dynamical anomalies such as the emergence of fractal-like structures in μ-space, q-exponential velocity correlations and superdiffusion, previously observed only for M(0)=1 initial conditions and pointing towards Tsallis thermodynamics scenario, are always present when the initial magnetization is greater than zero. Conversely, starting with M(0)=0, the dynamics produces a very peculiar kind of QSS, different from all the other cases. A possible explanantion of this different behavior could be the sudden initial quenching characterizing, with different intensities, the dynamics for finite initial magnetization. This fast relaxation is absent only for M(0)=0, which represents a limiting case. In general our results show that anomalous behavior is more ubiquitous than previously supposed.

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